Introduction

- Sequences are ordered lists of elements.
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, ...
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Sequences

Definition: A *sequence* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, ...} or {1, 2, 3, 4, ...}) to a set *S*.

- The notation a_n is used to denote the image of the integer *n*. We can think of a_n as the equivalent of *f(n)* where *f* is a function from {0,1,2,.....} to *S*. We call a_n a *term* of the sequence.

Sequences

Example: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

Geometric Progression

Definition: A *geometric progression* is a sequence of the form: $a, ar, ar^2, \ldots, ar^n, \ldots$

where the *initial term a* and the *common ratio r* are real numbers.

Examples:

1. Let $a = 1$ and $r = -1$. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let $a = 2$ and $r = 5$. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let $a = 6$ and $r = 1/3$. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Arithmetic Progression

Definition: A *arithmetic progression* is a sequence of the form: $a, a+d, a+2d, \ldots, a+nd, \ldots$

where the *initial term a* and the *common difference d* are real numbers.

Examples:

1. Let *a* = −1 and *d* = 4:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let *a* = 7 and *d* = −3:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let *a* = 1 and d = 2:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Strings

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).

- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by λ.
- The string *abcde* has *length* 5.

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Questions about Recurrence Relations

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 1,2,3,4,…. and suppose that $a_0 = 2$. What are a_1, a_2 and a_3?

[Here $a_o = 2$ is the initial condition.]

Solution: We see from the recurrence relation that

$$a_1 = a_o + 3 = 2 + 3 = 5$$
$$a_2 = 5 + 3 = 8$$
$$a_3 = 8 + 3 = 11$$

Questions about Recurrence Relations

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots.$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$ by:

- Initial Conditions: $f_0 = 0, f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6 .

Answer:

$$
\begin{aligned}
f_2 &= f_1 + f_0 = 1 + 0 = 1, \\
f_3 &= f_2 + f_1 = 1 + 1 = 2, \\
f_4 &= f_3 + f_2 = 2 + 1 = 3, \\
f_5 &= f_4 + f_3 = 3 + 2 = 5, \\
f_6 &= f_5 + f_4 = 5 + 3 = 8.
\end{aligned}
$$

Solving Recurrence Relations

- Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a *closed formula*.
- Various methods for solving recurrence relations will be covered in Chapter 8 where recurrence relations will be studied in greater depth.
- Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of induction (Chapter 5).

Iterative Solution Example

Method 1: Working upward, forward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

$$\vdots$$

$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example

Method 2: Working downward, backward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n\text{-}1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_n &= a_{n\text{-}1} + 3 \\
&= (a_{n\text{-}2} + 3) + 3 = a_{n\text{-}2} + 3 \cdot 2 \\
&= (a_{n\text{-}3} + 3\,) + 3 \cdot 2 \;= a_{n\text{-}3} + 3 \cdot 3 \\
&\quad . \\
&\quad . \\
&\quad . \\
&= a_2 + 3(n - 2) \;= (a_1 + 3) + 3(n - 2) \;= 2 + 3(n - 1)
\end{aligned}
$$

Financial Application

Example: Suppose that a person deposits $10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Let P_n denote the amount in the account after 30 years. P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$$

with the initial condition $P_o = 10,000$

Continued on next slide →

Financial Application

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\,P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Solution: Forward Substitution

$P_1 = (1.11)P_0$

$P_2 = (1.11)P_1 = (1.11)^2P_0$

$P_3 = (1.11)P_2 = (1.11)^3P_0$

$\vdots$

$P_n = (1.11)P_{n-1} = (1.11)^n\mathrm{P}_0 \quad = \quad (1.11)^n\, 10{,}000$

$P_n = (1.11)^n\, 10{,}000$ (Can prove by induction, covered in Chapter 5)

$P_{30} = (1.11)^{30}\, 10{,}000 = \$228{,}992.97$

Special Integer Sequences (*opt*)

- Given a few terms of a sequence, try to identify the sequence. Conjecture a formula, recurrence relation, or some other rule.
- Some questions to ask?
  - Are there repeated terms of the same value?
  - Can you obtain a term from the previous term by adding an amount or multiplying by an amount?
  - Can you obtain a term by combining the previous terms in some way?
  - Are they cycles among the terms?
  - Do the terms match those of a well known sequence?

Questions on Special Integer Sequences

Example 1: Find formulae for the sequences with the following first five terms: 1, ½, ¼, 1/8, 1/16

Solution: Note that the denominators are powers of 2. The sequence with $a_n = 1/2^n$ is a possible match. This is a geometric progression with $a = 1$ and $r = ½$.

Example 2: Consider 1,3,5,7,9

Solution: Note that each term is obtained by adding 2 to the previous term. A possible formula is $a_n = 2n + 1.$ This is an arithmetic progression with $a = 1$ and $d = 2$.

Example 3: 1, -1, 1, -1,1

Solution: The terms alternate between 1 and -1. A possible sequence is $a_n = (-1)^n$. This is a geometric progression with $a = 1$ and $r = -1$.

Useful Sequences

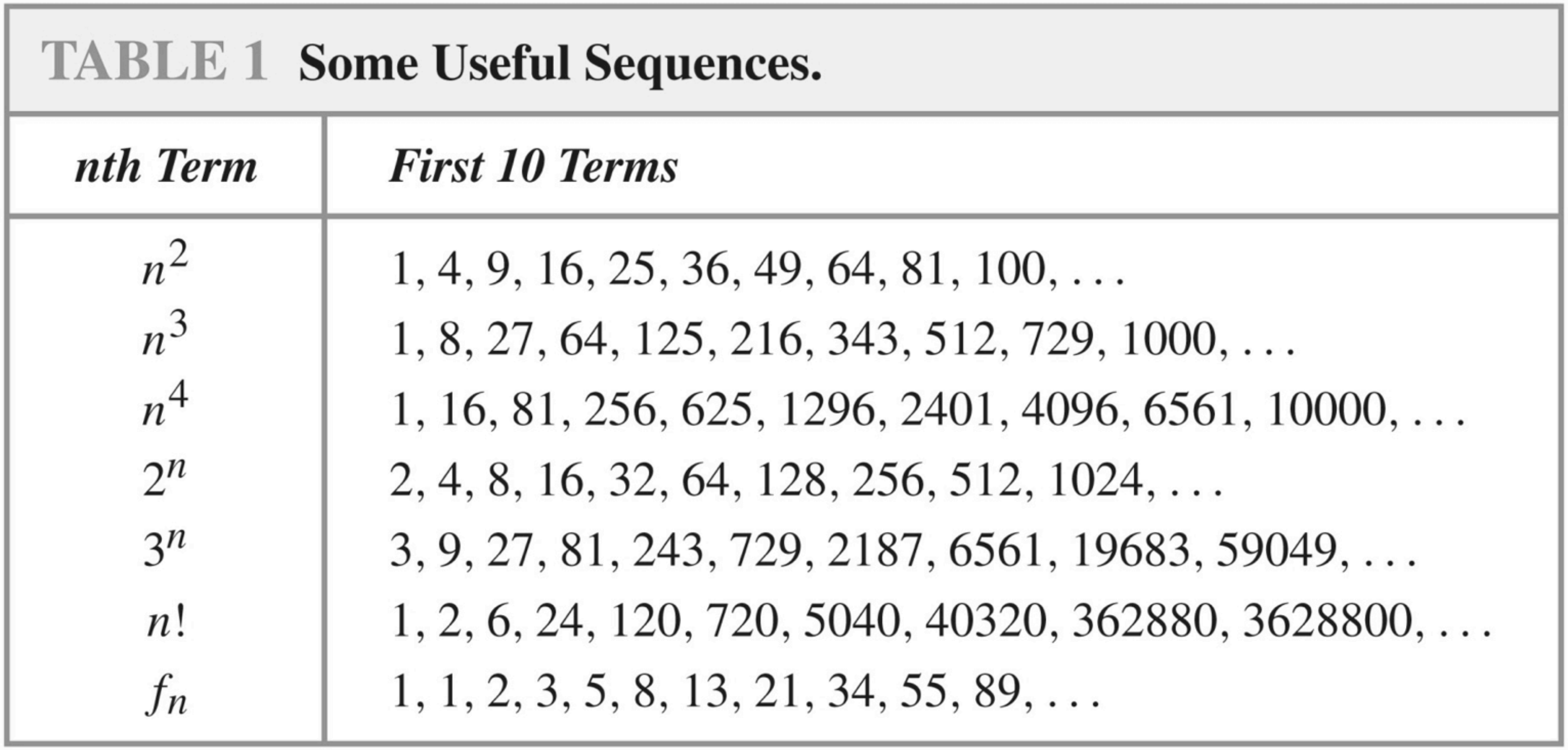


TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Guessing Sequences

Example: Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

Solution: Note the ratio of each term to the previous approximates 3. So now compare with the sequence 3^n. We notice that the nth term is 2 less than the corresponding power of 3. So a good conjecture is that $a_n = 3^n - 2$.

Summations

- Sum of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$
- The notation:

$$\sum_{j=m}^{n} a_j \quad \textstyle\sum_{j=m}^{n} a_j \quad \sum_{m \leq j \leq n} a_j$$

represents

$$a_m + a_{m+1} + \cdots + a_n$$

- The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit m* and ending with its *upper limit n*.

Summations

- More generally for a set *S*:

$$\sum_{j\in S} a_j$$

- **Examples**:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_0^n r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_1^\infty \frac{1}{i}$$

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j\in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Product Notation

- Product of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$
- The notation:

$$\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{m \le j \le n} a_j$$

represents

$$a_m \times a_{m+1} \times \cdots \times a_n$$

Geometric Series

Sums of terms of geometric progressions

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

Proof: Let $S_n = \sum_{j=0}^{n} ar^j$

To compute S_n, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$rS_n = r\sum_{j=0}^{n} ar^j$$

$$= \sum_{j=0}^{n} ar^{j+1}$$

Continued on next slide →

Geometric Series

$$= \sum_{j=0}^{n} ar^{j+1}$$ From previous slide.

$$= \sum_{k=1}^{n+1} ar^{k}$$ Shifting the index of summation with *k* = *j* + 1.

$$= \left(\sum_{k=0}^{n} ar^{k} \right) + (ar^{n+1} - a)$$ Removing *k* = *n* + 1 term and adding *k* = 0 term.

$$= S_n + (ar^{n+1} - a)$$ Substituting *S* for summation formula

$$\therefore \qquad rS_n = S_n + (ar^{n+1} - a)$$

$$S_n = \frac{ar^{n+1} - a}{r - 1}$$ if r ≠1

$$S_n = \sum_{j=0}^{n} ar^{j} = \sum_{j=0}^{n} a = (n+1)a$$ if r = 1

Some Useful Summation Formulae

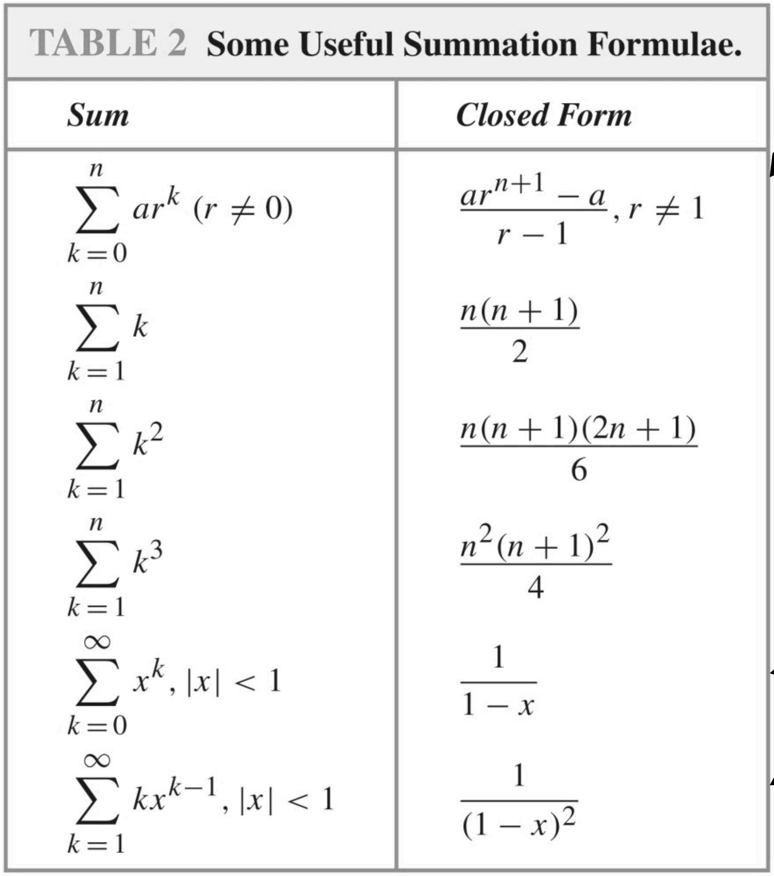


TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1 - x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1 - x)^2}$

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proofs use calculus

Cardinality

Definition: The *cardinality* of a set A is equal to the cardinality of a set B, denoted

$$|A| = |\mathrm{B}|,$$

if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from A to B.

- If there is a one-to-one function (*i.e.*, an injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.
- When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Cardinality

- **Definition**: A set that is either finite or has the same cardinality as the set of positive integers (**Z**$^+$) is called *countable*. A set that is not countable is *uncountable*.
- The set of real numbers **R** is an uncountable set.
- When an infinite set is countable (*countably infinite*) its cardinality is $\aleph_0$ (where $\aleph$ is aleph, the 1st letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

Showing that a Set is Countable

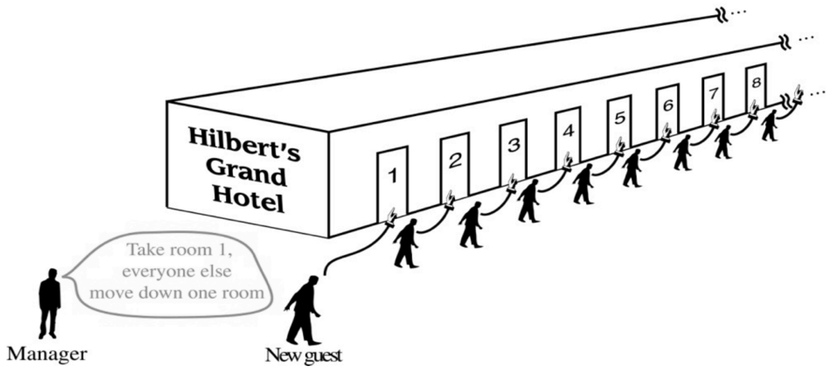
- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- The reason for this is that a one-to-one correspondence *f* from the set of positive integers to a set *S* can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n), \ldots$

Hilbert's Grand Hotel

David Hilbert

The Grand Hotel (example due to David Hilbert) has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

Explanation: Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room *n* to Room *n* + 1, for all positive integers *n*. This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.



The hotel can also accommodate a countable number of new guests, all the guests on a countable number of buses where each bus contains a countable number of guests (see exercises).

Showing that a Set is Countable

Example 1: Show that the set of positive even integers E is countable set.

Solution: Let $f(x) = 2x$.

$$\begin{tikzcd}
1 \arrow[d, leftrightarrow] & 2 \arrow[d, leftrightarrow] & 3 \arrow[d, leftrightarrow] & 4 \arrow[d, leftrightarrow] & 5 \arrow[d, leftrightarrow] & 6 \arrow[d, leftrightarrow] & \ldots\ldots \\
2 & 4 & 6 & 8 & 10 & 12 & \ldots\ldots
\end{tikzcd}$$

Then f is a bijection from **N** to E since f is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$. ◄

Showing that a Set is Countable

Example 2: Show that the set of integers **Z** is countable.

Solution: Can list in a sequence:

$0, 1, -1, 2, -2, 3, -3, \ldots\ldots\ldots$

Or can define a bijection from **N** to **Z**:

- When n is even: $f(n) = n/2$
- When n is odd: $f(\mathrm{n}) = -(n-1)/2$ ◀

The Positive Rational Numbers are Countable

- **Definition**: A *rational number* can be expressed as the ratio of two integers p and q such that $q \neq 0$.
  - ¾ is a rational number
  - √2 is not a rational number.

Example 3: Show that the positive rational numbers are countable.

Solution:The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \ldots$$

The next slide shows how this is done. →

The Positive Rational Numbers are Countable

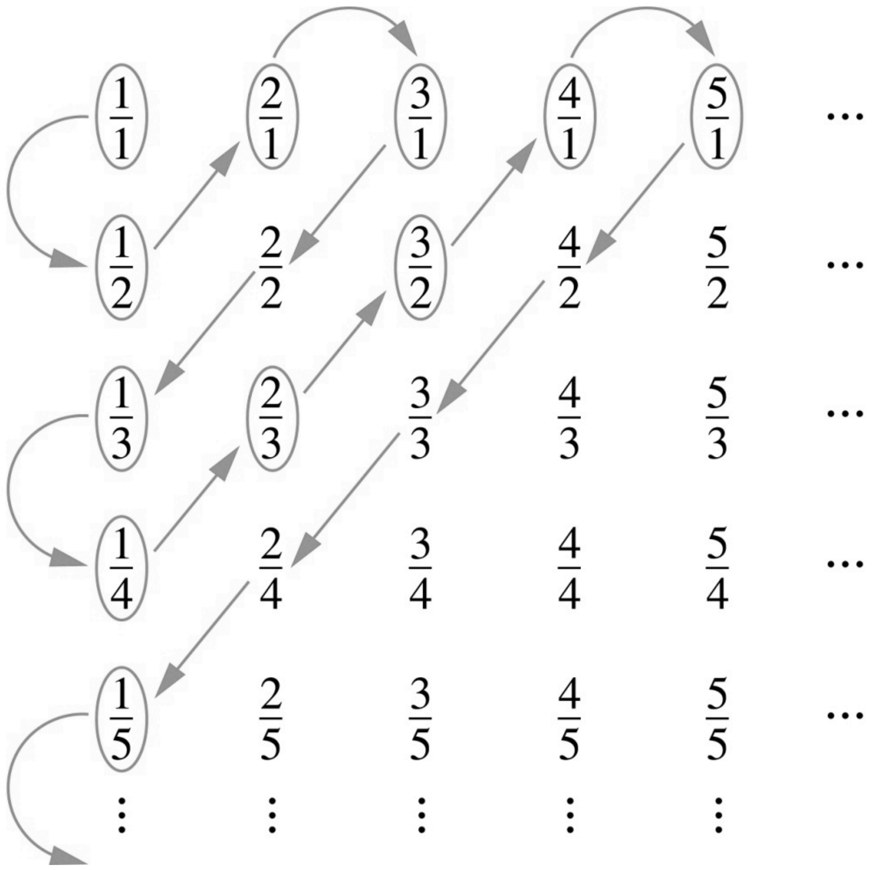
First row $q = 1$.
Second row $q = 2$.
etc.

Constructing the List

First list p/q with $p + q = 2$.
Next list p/q with $p + q = 3$

And so on.

Terms not circled are not listed because they repeat previously listed terms



1, ½, 2, 3, 1/3,1/4, 2/3, ….

Strings

Example 4: Show that the set of finite strings S over a finite alphabet A is countably infinite.

Assume an alphabetical ordering of symbols in A

Solution: Show that the strings can be listed in a sequence. First list

1. All the strings of length 0 in alphabetical order.
2. Then all the strings of length 1 in lexicographic (as in a dictionary) order.
3. Then all the strings of length 2 in lexicographic order.
4. And so on.

This implies a bijection from **N** to S and hence it is a countably infinite set.

The set of all Java programs is countable.

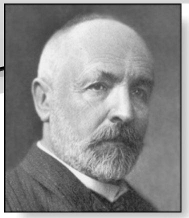
Example 5: Show that the set of all Java programs is countable.

Solution: Let *S* be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:

- Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program.)
- If the compiler says YES, this is a syntactically correct Java program, we add the program to the list.
- We move on to the next string.

In this way we construct an implied bijection from **N** to the set of Java programs. Hence, the set of Java programs is countable.

The Real Numbers are Uncountable



Georg Cantor
(1845-1918)

Example: Show that the set of real numbers is uncountable.

Solution: The method is called the Cantor diagnalization argument, and is a proof by contradiction.

1. Suppose **R** is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable - an exercise in the text).
2. The real numbers between 0 and 1 can be listed in order $r_1, r_2, r_3, \ldots$.
3. Let the decimal representation of this listing be
$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}\ldots \\ &\vdots \end{aligned}$$
4. Form a new real number with the decimal expansion $r = .r_1r_2r_3r_4\ldots$
where $r_i = 3 \text{ if } d_{ii} \neq 3$ and $r_i = 4 \text{ if } d_{ii} = 3$
5. r is not equal to any of the $r_1, r_2, r_3, \ldots$ Because it differs from r_i in its *i*th position after the decimal point. Therefore there is a real number between 0 and 1 that is not on the list since every real number has a unique decimal expansion. Hence, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.
6. Since a set with an uncountable subset is uncountable (an exercise), the set of real numbers is uncountable.

Computability

- **Definition**: We say that a function is **computable** if there is a computer program in some programming language that finds the values of this function. If a function is not computable we say it is **uncomputable**.
- There are uncomputable functions. We have shown that the set of Java programs is countable. There are uncountably many different functions from a particular countably infinite set (i.e., the positive integers) to itself. Therefore there must be uncomputable functions.